Separation Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713708471

Aspects off One-Dimensional Filtration

D. E. Smiles[a]; J. M. Kirby[a]
[a] CSIRO DIVISION OF SOILS CANBERRA, AUSTRALIA

To cite this Article Smiles, D. E. and Kirby, J. M.(1987) 'Aspects off One-Dimensional Filtration', Separation Science and Technology, 22: 5, 1405 — 1423
To link to this Article: DOI: 10.1080/01496398708058407
URL: http://dx.doi.org/10.1080/01496398708058407

SEPARATION SCIENCE AND TECHNOLOGY, **22**(5), pp. 1405-1423, 1987

Aspects of One-Dimensional Filtration

D. E. SMILES and J. M. KIRBY

CSIRO DIVISION OF SOILS
CANBERRA, AUSTRALIA

Abstract

In many circumstances, filtration of two-phase (solid–liquid) systems may be described by theory based on Darcy's law and an appropriate continuity equation. In following this approach it is important to recognize that, since both phases are generally in motion relative to an observer, Darcy's law describes the flow of the liquid relative to the solid particles in response to a space gradient of potential (head). It also emerges that analysis is often simplified by recasting the flow equations in a coordinate system based on the distribution of the solid component of the system. The theory requires that relations between the water content and (a) the water potential and (b) the permeability are well defined. Neither functional permits à priori prediction, although various formulas have been proposed in particular circumstances. This paper describes simply and directly the formulation of an appropriate material coordinate. It also provides experimental information relating to important aspects of filtration and the permeability of bentonite slurries subject to constant pressure filtration. It is shown that the one theory describes both filtration and expression, so the distinction between them is artificial.

INTRODUCTION

Theories of filtration and of consolidation have their origins in chemical engineering (*1–6*), in civil engineering (*7–11*), and in soil science (*12–17*). There are, however, difficulties which restrict application of theory in each discipline.

The first is the problem of semantics, and the use of different functional variables in the different disciplines. An example is provided by the use of water potential in soil physics to describe what in chemical engineering is the negative of the solid compressive pressure, or in civil engineering the negative of the effective stress. Another problem arises when empirical relations are used prematurely in the derivation of the flow equations appropriate to the processes under consideration. For example, the use of the Kozeny-Carman relation to relate the permeability (or hydraulic conductivity) to the void ratio (or volume fraction of water) of the system early in the derivation of a flow equation often greatly complicates the form of the equation and tends to obscure the principles on which it is based.

This paper addresses two issues:

(1) It derives the filtration/consolidation equation, taking as bases the continuity equations of the liquid and the solid, and Darcy's law. In the process the physical rationale for material coordinates which much simplify the solution of flow problems is set out, together with the reason for using volume fraction in physical space and void ratio in material space (*8*).

(2) It provides comment on recent developments in filtration (and consolidation) theory and, in particular, on implications in the literature (*3, 4*) that the early (filtration) and later stages (expression) of constant pressure dewatering are somehow different.

Reference is made to a series of experiments in which volumes of bentonite slurry are filtered by the imposition of constant pressure in a filter cell at the base of which is a membrane that permits escape of liquid but not solid.

The theory is developed here quite specifically for flow in a two-phase system for which

$$\theta_s + \theta_w = 1 \tag{1}$$

where θ_s and θ_w are volume fractions of solid and water, respectively.

THEORY

In a two-phase system during one-dimensional nonsteady flow, equations of continuity may be written for the water (Eq. 2) and for the solid (Eq. 3), viz.,

$$\left(\frac{\partial \theta_w}{\partial t}\right)_z = -\left(\frac{\partial F_w}{\partial z}\right)_t \tag{2}$$

$$\left(\frac{\partial \theta_s}{\partial t}\right)_z = -\left(\frac{\partial F_s}{\partial z}\right)_t \tag{3}$$

In these equations F_w and F_s are the volume fluxes of water and solid, respectively, relative to an external observer, and z and t are distance and time, respectively.

During flow involving water content change, both the water and the solid are in motion, so the flux of water relative to an observer has a component, u, relative to the solid particles, and a "convective" component associated with the moving particles. Thus, F_w may be written

$$F_w = u + \theta_w F_s/\theta_s = u + \vartheta F_s \tag{4}$$

In Eq. (4), F_s/θ_s is the average velocity of the solid and $\theta_w/\theta_s = \vartheta$ is the moisture ratio (volume of water per unit volume of solid). In a saturated system, ϑ is identical to the void ratio, e: the distinction is maintained to anticipate situations where air may enter the system and $\vartheta < e$ (*13*).

If we now substitute for F_w from Eq. (4) in Eq. (2), and also substitute $\vartheta\theta_s$ for θ_w, we obtain

$$\left(\frac{\partial(\theta_s\vartheta)}{\partial t}\right)_z = -\left(\frac{\partial u}{\partial z}\right)_t - \left(\frac{\partial(\vartheta F_s)}{\partial z}\right)_t \tag{5}$$

Differentiation by parts followed by the elimination of two terms using Eq. (3), and division by θ_s then yield

$$\left(\frac{\partial \vartheta}{\partial t}\right)_z = -\left(\frac{1}{\theta_s}\right)\left(\frac{\partial u}{\partial z}\right)_t - \left(\frac{F_s}{\theta_s}\right)\left(\frac{\partial \vartheta}{\partial z}\right)_t \tag{6}$$

Equation (6) provides a basis for an Eulerian analysis of unsteady flow problems expressed in terms of the space coordinate z. Such an approach was developed by Philip (*12*) and more recently by Wakeman (*18*). Alternatively, a Lagrangian analysis may be used if we recognize that the left-hand side of Eq. (6), together with the second term on the right, represent the differential of ϑ following the motion of the solid (e.g., Refs. *13, 14*), that is,

$$\left(\frac{\partial \vartheta}{\partial t}\right)_z + \left(\frac{F_s}{\theta_s}\right)\left(\frac{\partial \vartheta}{\partial z}\right)_t = \left(\frac{\partial \vartheta}{\partial t}\right)_m \tag{7}$$

with $m(z,t)$ a material coordinate defined by the equations

$$\partial m/\partial z = \theta_s \tag{8}$$

and

$$\partial m/\partial t = -F_s \tag{9}$$

so that

$$dm = \theta_s dz - F_s dt \tag{10}$$

Note that Eq. (8) and Eq. (9) satisfy the continuity Eq. (3) for the solid.

Substitution of Eq. (7) in Eq. (6) and the use of Eq. (8) yields the continuity equation for the water in material space, viz.,

$$\left(\frac{\partial \vartheta}{\partial t}\right)_m = -\left(\frac{\partial u}{\partial m}\right)_t \tag{11}$$

Further development of the theory concentrates on Eq. (11) and in particular on the laws of flow necessary to define u.

Darcy's Law for Colloidal Systems

In the system we describe, Darcy's law describes the volume flux of the water *relative to the particles* in response to a *space gradient* of piezometric head (*19*). Here we use the total potential, Φ, of the water rather than the piezometric head, although the two are simply related (as we shall show). Darcy's law then becomes

$$u = -k(\vartheta)\nu^{-1}(\partial \Phi/\partial z) \tag{12}$$

In Eq. (12), u has units (m/s), ν is the kinematic viscosity of water (m^2/s), and $k(\vartheta)$ is the water content dependent permeability. If Φ is expressed as energy per unit mass of water, with SI units J/kg, then the permeability takes units of m^2. The form of $k(\vartheta)$ in systems that change their volume with ϑ is well known (*9, 10*), and Carman (*20*), in particular, explored methods for its prediction. The development of this theory assumes that $k(\vartheta)$ is well defined.

Total Potential of Water

In a two-phase swelling material, the total potential of the water is given by

$$\Phi = \Psi(\vartheta) + \Omega + gz \tag{13}$$

In Eq. (13), g is the acceleration due to gravity so gz is the gravitational potential of water at z relative to a convenient datum, Ω is the overburden potential, and $\Psi(\vartheta)$ is the water content dependent potential that arises as a result of interaction of the water with the solid surfaces and their geometry. $\Psi(\vartheta)$ is readily measured (*21*).

The overburden potential (*22, 23*) is defined by

$$\Omega = g\int_{z}^{z_1} \gamma_w dz + P \tag{14}$$

where γ_w is the wet specific gravity of the system, $z = z_1$ is its upper surface, and P is any normal surface load.

Combination of Eq. (13) and Eq. (14) yields

$$\begin{aligned}\Phi &= \Psi(\vartheta) + g\int_{z}^{z_1} \gamma_w dz + P + gz \\ &= p_w(z) + gz \end{aligned}\tag{15}$$

In Eq. (15), $p_w(z)$ is the water pressure measured with a manometer fitted with a membrane that permits passage of water but not solid. According to Eq. (15), $\Psi(\vartheta)$ is the negative of the "effective stress" of civil engineering theory (*24*) or the interparticle or solid compressive pressure of filtration theory (*25*).

Substitution of Eq. (13) in Eq. (12) and the inclusion of Eq. (14) then yields Darcy's law in the form

$$\begin{aligned}u &= -k(\vartheta)\nu^{-1}\left(\frac{\partial\Psi}{\partial z} + g(1 - \gamma_w)\right) \\ &= -\frac{k(\vartheta)}{\nu(1 + \vartheta)}\left(\frac{\partial\Psi}{\partial m} + g(1 - \gamma)\right)\end{aligned}\tag{16}$$

In Eq. (16), γ is the specific gravity of the solid component of the system, and we have used Eq. (8), noting that $\theta_s = (1 + \vartheta)^{-1}$, to derive the second equality. The group $k(\vartheta)\nu^{-1}(1 + \vartheta)^{-1}(s)$ plays the same role in material space as the hydraulic conductivity $k(\vartheta)\nu^{-1}(s)$ plays in physical space.

Equation of Unsteady Vertical Flow in a Two-Component System

A general equation of flow now follows if we substitute for u from Eq. (16) in Eq. (11):

$$\frac{\partial\vartheta}{\partial t} = \frac{\partial}{\partial m}\left(\frac{k(\vartheta)}{\nu(1+\vartheta)}\frac{\partial\Psi}{\partial m}\right) - g(\gamma - 1)\frac{\partial}{\partial m}\left(\frac{k(\vartheta)}{\nu(1+\vartheta)}\right) \tag{17}$$

which may be written as the nonlinear Fokker-Planck equation (*26, 27*)

$$\frac{\partial\vartheta}{\partial t} = \frac{\partial}{\partial m}\left(D(\vartheta)\frac{\partial\vartheta}{\partial m}\right) - E(\vartheta)\frac{\partial\vartheta}{\partial m} \tag{18}$$

in which the moisture diffusivity D is given by

$$D = \frac{k(\vartheta)}{\nu(1+\vartheta)}\frac{d\Psi}{d\vartheta} \tag{19}$$

and the coefficient E given by

$$E = g(\gamma - 1)\frac{d}{d\vartheta}\left(\frac{k(\vartheta)}{\nu(1+\vartheta)}\right) \tag{20}$$

The moisture diffusivity D (m^2/s) will be recognized as a coefficient of consolidation in civil engineering terms (*7–10*) and as an expression coefficient in filtration theory (*1, 3, 6*).

The coefficient E (m/s), which embodies the effects of gravity both directly and through the overburden component of potential, does not generally appear in filtration theory.

The fact that both D and E vary with ϑ complicates the solution of Eq. (20). There is, nevertheless, a substantial literature, in particular in soil physics and hydrology, devoted to analytical, quasi-analytical, and numerical methods of solving this equation subject to relevant conditions (e.g., Refs. *28–31*).

In practice, however, this full solution may be unnecessary for many materials and flow conditions for a material with, for example, $\gamma = 2.6$ and $\vartheta = 25$, the "effect of gravity" is, as we show later, substantially reduced. The flow process, in consequence, approximates more closely that of a "gravity-free" system for which Eq. (18) becomes the nonlinear diffusion equation

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial m}\left(D(\vartheta)\,\frac{\partial \vartheta}{\partial m}\right) \tag{21}$$

The importance of gravity also diminishes as the imposed pressure P_i is increased (cf. Eq. 15), and the major features of constant pressure filtration, which we use below, are well predicted by solutions to Eq. (21).

Experimental

The experiments illustrate basic principles of the approach and also permit comment on the relationship between the "early" and "later" stages of dewatering. Data also permit test of the Kozeny-Carman model for calculating the $k(\theta)$ relation of a slurry of Wyoming bentonite.

The experiments were performed in cylindrical pressure cells of cross-section $A = 11.34\ \text{cm}^2$ at the base of which a 0.45-µm filter membrane permitted escape of water, but not clay, to atmospheric pressure. The escaping water was collected on a top-weighing balance, so the outflow rate could be measured.

The clay had an initial water content, ϑ, of 37.5 and a particle specific gravity of 2.6.

Material Coordinate

The m-coordinate is determined by integrating Eq. (10), taking advantage of the fact that at the filter membrane ($z = 0$) the flux of solid, F_s, is zero. Thus

$$m(z,t) = \int_0^z \theta_s(\tilde{z},t)\,d\tilde{z} - \int_0^t F_s(0,\tilde{t})\,d\tilde{t} \tag{22}$$

$$= \int_0^z (1 + \vartheta)^{-1}\,dz \tag{23}$$

and m is therefore the cumulative volume of the solid, per unit area of cross section, measured away from the membrane.

In the experiments described here, a known volume, V, of wet clay is added to the filtration cell. Thus, the material length of the system is a constant M given by

$$M = V/A(1 + \vartheta_n) \tag{24}$$

in which ϑ_n (= 37.5) is the initial uniform liquid content of the clay.

Initial and Boundary Conditions

In material space the initial condition is

$$\vartheta = \vartheta_n\ (= 37.5), \qquad 0 < m \leqslant M, \qquad t = 0 \tag{25}$$

The boundary conditions, following the imposition of a constant pressure at $t = 0$, are (*15*)

$$\vartheta = \vartheta_0, \qquad m = 0, \qquad t > 0 \tag{26}$$

and

$$\partial\vartheta/\partial m = 0, \qquad m = M, \qquad t > 0 \tag{27}$$

with ϑ_0 the water content in equilibrium with the imposed pressure for $p_w = 0$ (cf. Eq. 15).

Two sets of experiments were performed. In the first, 10 ml samples of clay were subjected to various constant pressures equivalent to values of $\Psi/(\mathrm{J/kg})$ in the range $-1 > \Psi/(\mathrm{J/kg}) > -10^2$. During these experiments, the outflow was measured, and when equilibrium was judged to exist, the equilibrium water content was determined by oven drying (at 105°C) the clay remaining in the pressure cell.

Figure 1 shows equilibrium $\Psi(\vartheta)$ data from these experiments.

In addition, a subset was performed at a particular pressure (equivalent to $\Psi = -64$ J/kg). In this group, different initial volumes of the clay slurry were used. For each experiment the (constant) material length, M, was calculated using Eq. (24).

Figure 2 shows cumulative outflow, i, graphed as a function of $t^{1/2}$ for these experiments. The data are in the reduced forms (i/M) and $(t^{1/2}/M)$ to eliminate the effect of V.

In the second series of experiments, longer columns of clay were used, and this series was terminated while the water content at the distal end remained unchanged at $\vartheta = 37.5$. The spatial distributions of water and

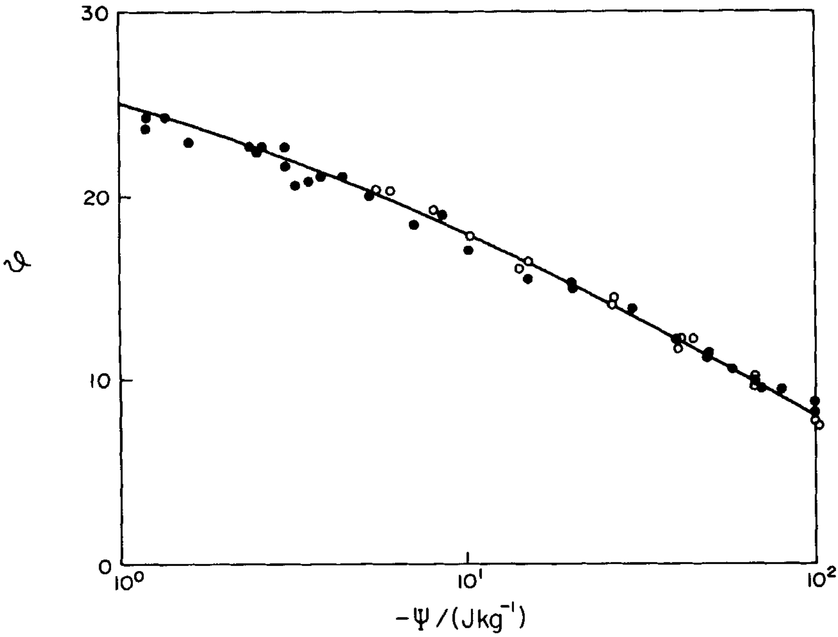

FIG. 1. The characteristic equilibrium relation between water content, ϑ, and water potential, Ψ, for the clay used in the experiments described herein.

clay were determined by destructive sectioning and oven drying. All experiments in this series were submitted to a pressure equivalent to $\Psi/(\mathrm{J/kg}) = -64$ but were terminated at different times t.

For this set of experiments the Boltzmann variable, $\lambda = mt^{-1/2}$, then may be used to eliminate m and t from both the flow Eq. (21) and the conditions for its solution (Eqs. 25, 26) and, by implication, $\vartheta(\lambda)$ is the unique solution of this set of equations. So, if Eq. (21) is valid and Eqs. (25) and (26) are realized experimentally, then the data from this second series of experiments must be unique if graphed as ϑ vs λ.

Figure 3 shows that these data so plotted are indeed effectively unique.

DISCUSSION

In what follows, we need details of the solution of Eq. (21) subject to Eqs. (25) and (26). Full theory is set out elsewhere (*28*) but, briefly, the

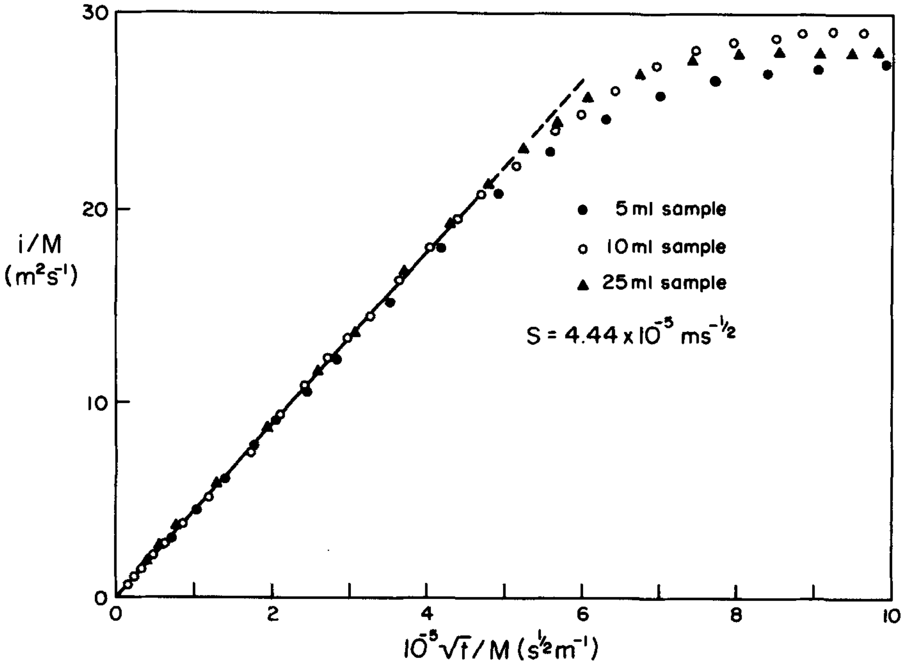

FIG. 2. Cumulative outflow, i, graphed as a function of $t^{1/2}$ for columns of clay of initial water content $\vartheta = 37.5$ and material length $M = 1.15 \times 10^{-4}$ m (●), 2.3×10^{-4} m (○), and 5.73×10^{-4} m (▲). Each column was subject to a constant pressure of 64 kN/m². The effect of M is eliminated by use of the variables i/M and $t^{1/2}/M$.

introduction of λ in the equations followed by integration of the transformed Eq. (21) using Eq. (25) yields

$$D(\vartheta)\frac{d\vartheta}{d\lambda} = -\int_{\vartheta_n}^{\vartheta} \frac{\lambda}{2} d\vartheta \tag{28}$$

and then using Eq. (26)

$$D(\vartheta)\frac{d\vartheta}{d\lambda_{\lambda=0}} = -\int_{\vartheta_n}^{\vartheta_0} \frac{\lambda}{2} d\vartheta = -\frac{S(\vartheta_0,\vartheta_n)}{2} \tag{29}$$

Equation (28) provides the basis for calculating $\vartheta(\lambda)$ if $D(\vartheta)$ is known (*28–31*), or a method (*28*) for calculating $D(\vartheta)$ from $\vartheta(\lambda)$.

Figure 4 shows $D(\vartheta)$ so calculated for the data of Fig. 3.

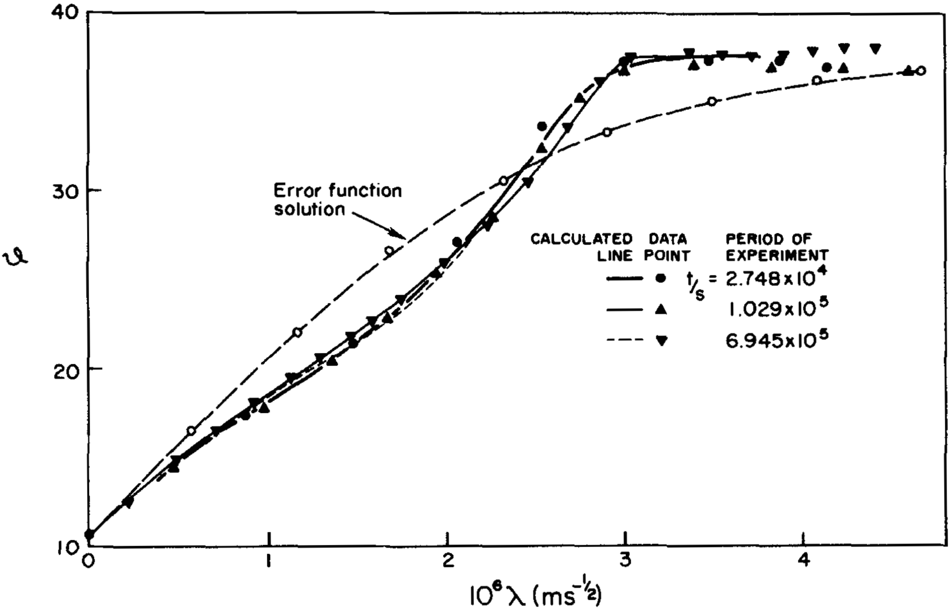

FIG. 3. Profiles, observed during filtration, of water content, ϑ, graphed as a function of material distance, m, divided by $t^{1/2}$, for clay with initial water content $\vartheta = 37.5$, and subject to pressure of 64 $k\mathrm{N/m}^2$. Also shown is the error function solution of Eq. (21) subject to relevant conditions and which yields correct integral behavior for the system.

Equation (29), differentiated with respect to x and integrated with respect to t, yields

$$-i = \int_0^t u_{m=0} dt = S(\vartheta_0,\vartheta_n)t^{1/2} \tag{30}$$

This equation follows if we recognize that within the gravity-free analysis (for which Eq. 21 is appropriate), combination of Eq. (12) and Eq. (19) yields

$$u = -D(\vartheta)\frac{\partial\vartheta}{\partial x} \tag{31}$$

Within this framework, we make the following comments:

(1) The hydraulic properties of this material are fully defined by the data of Figs. 1 and 4. Neither relation is predictable, but both are macroscopic and readily measurable. They do not depend on any

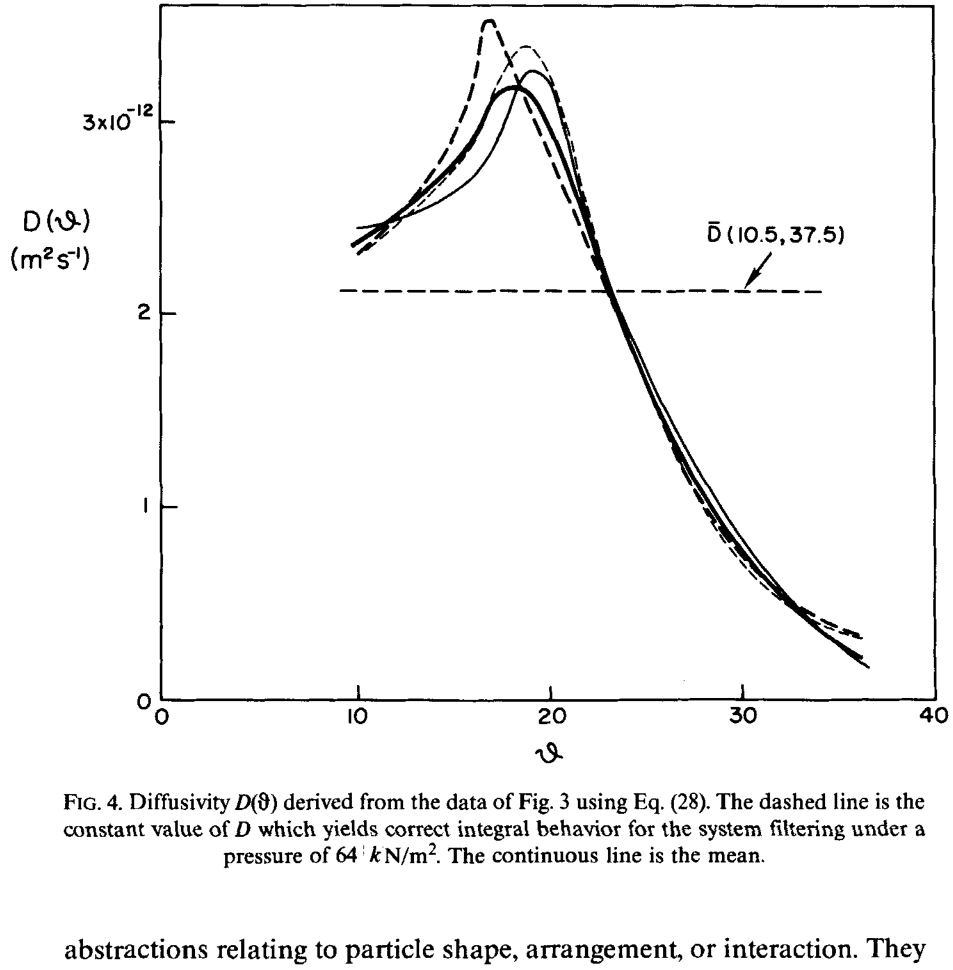

FIG. 4. Diffusivity $D(\vartheta)$ derived from the data of Fig. 3 using Eq. (28). The dashed line is the constant value of D which yields correct integral behavior for the system filtering under a pressure of 64 kN/m^2. The continuous line is the mean.

abstractions relating to particle shape, arrangement, or interaction. They do, of course, represent the appropriate test of models so based.

(2) Figure 2 reveals cumulative outflow linear with regard to $t^{1/2}$ for the "early stages" of the filtration process, as Eq. (30) requires. Furthermore, the slope of the curve equals the integral in Eq. (29) evaluated with the data of Fig. 3, as it must. On closer comparison, it emerges that the reduced time $t^{1/2}/M$, for which Fig. 2 data are linear, exceeds by a factor of ~1.66 the value of $1/\lambda$ in Fig. 3 where $\vartheta \to \vartheta_n$. Thus, $t^{1/2}$ behavior continued ~2.75 times longer than the time when the water content at the distal end commenced to decrease. "Square-root-of-time" behavior therefore reveals no indication of a time when the process of "filtration" becomes one of "expression" or "consolidation."

Physically, this somewhat surprising behavior arises because of the "shape" of the $D(\vartheta)$ relation in Fig. 4 and the fact that flow is related to the integral of $D(\vartheta)$ but weighted toward the outflow surface where $\vartheta = \vartheta_0$ and $D(\vartheta)$ is relatively great (*32*).

Specifically, for the semi-infinite case, it can be shown (*30*) that

$$S^2(\vartheta_0,\vartheta_n) = 2\int_{\vartheta_n}^{\vartheta_0} \frac{(\vartheta - \vartheta_n)D(\vartheta)}{F(\vartheta,\vartheta_0,\vartheta_n)}\,d\vartheta$$

$$= \frac{4}{\pi}(\vartheta_0 - \vartheta_n)^2\bar{D}(\vartheta_0,\vartheta_n) \tag{32}$$

Thus the appropriate mean value of $D(\vartheta)$ is given by

$$\bar{D}(\vartheta_0,\vartheta_n) = \frac{\pi}{2}(\vartheta_0 - \vartheta_n)^2\int_{\vartheta_n}^{\vartheta_0} \frac{(\vartheta - \vartheta_n)D(\vartheta)}{F(\vartheta,\vartheta_n,\vartheta_0)}\,d\vartheta \tag{33}$$

In these equations, F is the flux-concentration relation (*29*), i.e., the characteristic ratio of flux at $\vartheta = \vartheta$, to that at $\vartheta = \vartheta_0$; and $\bar{D}(\vartheta_0,\vartheta_n)$ is the appropriate mean value of $D(\vartheta)$. The second equality in Eq. (32) is the exact solution of the linear form of Eq. (21) subject to Eqs. (25) and (26) (*33*).

If now we assume (reasonably) (*32*) that for filtration

$$\theta < F < \theta^{2-\pi/2} \tag{34}$$

where

$$\theta = (\vartheta - \vartheta_n)/(\vartheta_0 - \vartheta_n) \tag{35}$$

it follows that

$$\int_{\vartheta_n}^{\vartheta_0} D(\vartheta)d\vartheta < \frac{2\bar{D}(\vartheta_0,\vartheta_n)}{\pi(\vartheta_0 - \vartheta_n)} < \int_{\vartheta_n}^{\vartheta_0} \theta^{\pi/2-1}D(\vartheta)d\vartheta \tag{36}$$

where the "shape" of $D(\vartheta)$ and the effect of θ in the right-hand integral conspire to weight $\bar{D}$ toward a value close to that at ϑ_0. By implication then, flow is relatively insensitive to what is happening at the distal end of the column.

(3) The solution of the linear form of Eq. (21). For the data of these experiments, $\bar{D} = 2.12 \times 10^{-12}$ m²/s yields integral behavior which corresponds with observation shown in Fig. 2. In this sense, then, the solution

of the linear form of Eq. (21) subject to Eqs. (25) and (26) is exact. The penalty for linearization, however, is shown in Fig. 3, where the linear solution (*33*) is shown as the smooth (error function) curve. Clearly, this solution gives an incorrect prediction of the spatial distribution of the solid and the water, although its "integral" behavior corresponds to reality.

(4) The relationship of $D(\vartheta)$ defined for material space with the corresponding coefficient in "x-space." Equation (5) was developed (*12, 14*) to yield an equation corresponding to Eq. (21) in x-space. In that space the coefficient corresponding to $D(\vartheta)$ is $D(\theta)$. The two are related according to

$$D(\vartheta) = D(\theta)/(1 + \vartheta)^3 \tag{37}$$

For the material we deal with, the variation in $D(\theta)$ is about the same as that of $D(\vartheta)$ in the corresponding moisture content ranges, so it might appear to be sensible to work in real space. In fact, however, the arithmetic is more complicated (*12*), and methods for solving Eq. (21) (or Eq. 18) are so well-developed that the material approach is preferable (cf. Refs. *12* and *14*).

(5) Equation (19), together with the data of Figs. 1 and 4, permits us to calculate the $k(\vartheta)$ relation for this clay. The result of this calculation is shown as Fig. 5. These data may then be compared with the corresponding functional calculated using the approach of Kozeny-Carman in which k is taken to be proportional to the cross-sectional area available for flow, θ_w, and to the square of the characteristic length of colloid particle separation, $(\theta_w/\theta_s)^2$, i.e.,

$$k \propto \theta_w^3/\theta_s^2 \tag{38}$$

From Fig. 5 it is evident that the Kozeny-Carman approximation does not provide a good basis for prediction of $k(\vartheta)$ for this clay.

(6) The importance of gravity. Formally, the "early stages" of filtration in a vertical cell require, for their description, the solution of Eq. (18). The data presented here justify the use of the simpler Eq. (21), but it is still useful to define conditions under which the full solution is required; particularly since the problem is raised in a formal sense by, for example, Gibson et al. (*8*). A useful, order-of-magnitude estimate of the time for which a gravity-free analysis is appropriate is provided by Philip (*28*) and used by Smiles (*34*).

Briefly, the solution of Eq. (18) subject to Eqs. (24) and (25) can be written as the series

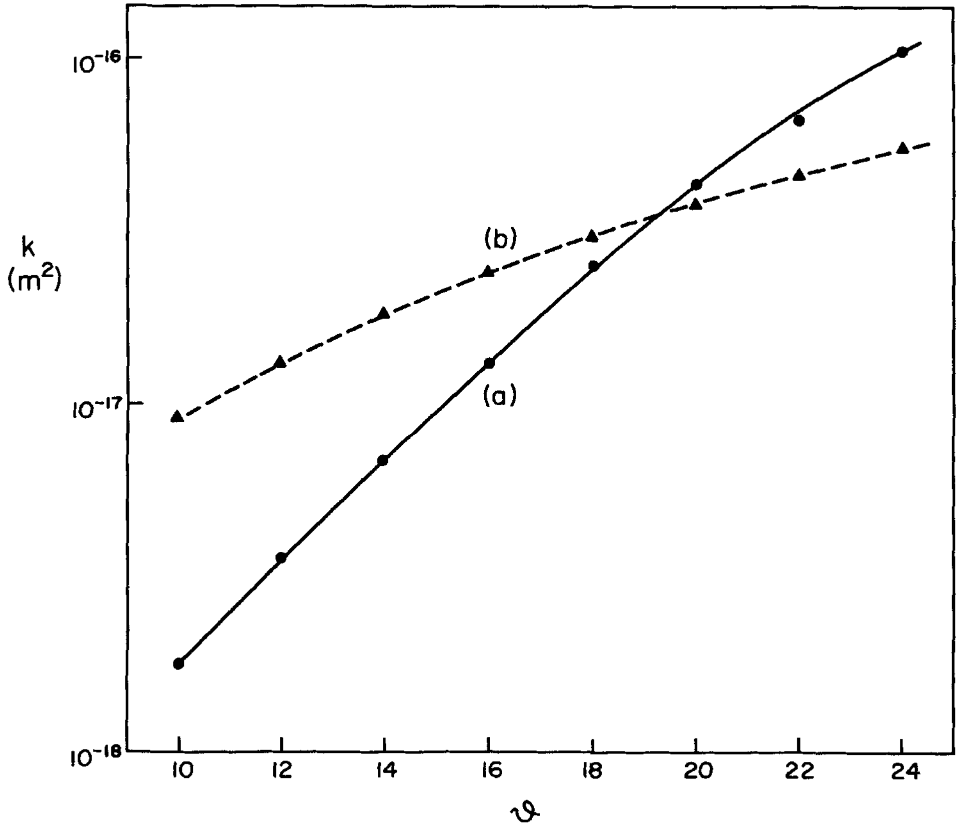

FIG. 5. Graphs showing k as a function of ϑ determined (a) using Fig. 1 and Fig. 4 and Eq. (19); and (b) the approach of Kozeny-Carman matched to (a) at $\vartheta = 19$. Evidently the Kozeny-Carman approach is unreliable for this material.

$$-i(t) = St^{-1/2} + \alpha t + \beta t^{1/2} + \cdots \tag{39}$$

in which S is the solution to the gravity-free Eq. (21) and the subsequent terms represent the effect of gravity. A comparison of the first and second terms of the series yields the required estimate. Thus, if t_g is the time for which $St^{1/2} \gg 100\alpha t$ and we set $\alpha = \frac{1}{2}(g(\gamma - 1)k(\vartheta_n)/\nu(1 + \vartheta_n))$ (*33*), then for the material described here,

$$\begin{aligned} t_g &= 10^{-4}(S/\alpha)^2 \\ &= 2.239 \times 10^7 \text{ s} \end{aligned} \tag{40}$$

which would correspond to a column of initial material length 1.9×10^{-2} m, or actual length 0.64 m.

It should be emphasized here that it is not generally true that gravity effects can be neglected, and certainly data for "red mud" suggest that filtration (or sedimentation) in that material requires solution for appropriate conditions of the full Eq. (18) (*35, 36*). Filtration for longer periods of time require formal solution of Eq. (18). Solutions of this equation subject to conditions (24) and (25) are available (*30, 37, 38*), and immediate recourse to linearization (e.g., Ref. *8*) is unnecessary and often misleading since linearization, as we show above, produces correct integral behavior but carries the penalty that the profile shapes can be substantially in error.

(7) The requirement that the water content remains unchanged at the distal end of the column. It must be admitted that the analytical and quasi-analytical solutions alluded to here do not apply to columns once $\vartheta(M) < \vartheta_n$, and the approach to equilibrium must then be handled numerically. This problem is not of great moment, however, and matching of the quasi-analytical approach for early stages of outflow (where the numerical procedures are relatively inefficient), with the efficient numerical calculation of the final stages (of expression), appears to provide a rational compromise. This procedure will be described elsewhere.

(8) Conversion of data from material to "real" space. Various important aspects of filtration can be derived by integration or differentiation of $\vartheta(\lambda)$ and manipulation of Eq. (4). Here it suffices to demonstrate the conversion from m-space to z-space for the data of Fig. 3. For this data set it follows from Eq. (10) that the variables λ $(= mt^{-1/2})$ and χ $(= zt^{-1/2})$ are related according to $\chi = \int_0^\lambda (1 + \vartheta) d\lambda$.

Figure 6 shows $\lambda(\chi)$ calculated using the data of Fig. 3. It will be noted that since λ, in a reduced sense, is the cumulative volume of solid measured away from $z = 0$, Fig. 6 shows the spatial distribution of solid volume in the filter cake and its evolution in time. The dashed curve in Fig. 6 shows the error incurred if the problem is linearized, and the error function "solution" of the problem is used as a basis for calculation with this set of data.

(9) Other features of filtration. It was not the purpose of this paper to set out in full the details that emerge from this approach to filtration. It is important to note, however, that other aspects of the process are readily derived from $\vartheta(\lambda)$. For example, the distribution of water potential is readily determined from Figs. 3 and 1, while the water pressure p_w may then be calculated using Eq. (15).

In addition, F_s and F_w can be determined by noting, from Eq. (30) and the continuity requirement, that

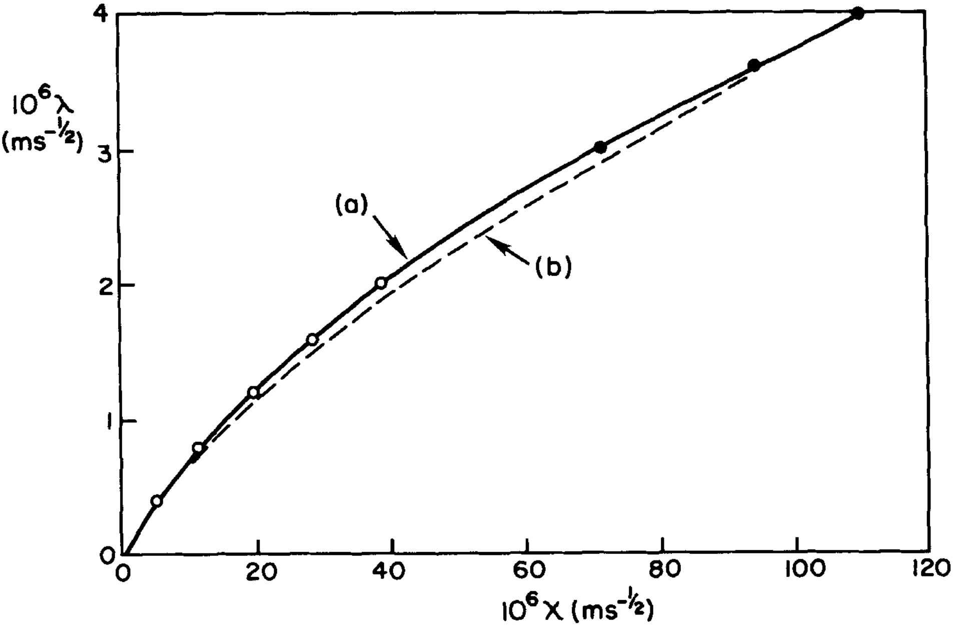

FIG. 6. Graph relating the material coordinate m (in reduced form $\lambda = mt^{-1/2}$) to "real" space a (in the form $\chi = zt^{-1/2}$). This graph also reveals the cumulative distribution of solid in real space because of the definition of m (Eq. 23).

$$F_w + F_s = -i = St^{1/2} \tag{41}$$

whence, from Eq. (4),

$$F_s = (St^{1/2} - u)/(1 + \vartheta) \tag{42}$$

and

$$F_w = (\vartheta St^{1/2} + u)/(1 + \vartheta) \tag{43}$$

in which the Darcy flux u is given by

$$u = -t^{1/2} \int_{\vartheta_n}^{\vartheta} (\lambda/2)\, d\vartheta \tag{44}$$

which is derived from Eqs. (28) and (31).

CONCLUDING REMARKS

We have described simply and directly the formulation of an equation describing one-dimensional filtration of a dispersion of solid particles in liquid, using a membrane impermeable to the solid.

The approach, which is based on Darcy's law and continuity equations for the liquid and the solid, depends on the existence of functionals relating the permeability and the liquid potential to the liquid content of the suspension. Although neither relation permits precise prediction, both are readily measured.

If the problem is cast in material coordinates (which satisfy the continuity equation for the solid component), the flow equation takes the form of a nonlinear Fokker-Planck equation which reduces to a nonlinear diffusion equation in circumstances where gravity effects may be neglected.

Methods of solution of these equations, without recourse to linearization, are well known.

Data are presented which yield no evidence to justify treating consolidation or filtration as physically different processes.

Acknowledgments

The authors wish to acknowledge the very helpful discussions with Dr P. A. C. Raats which led to the explanation of the material coordinate presented in the Theory section. This derivation is a particular example of a theory of mixtures discussed by Raats (*39*), and which might be useful for readers interested in a more general dissertation on that subject.

REFERENCES

1. K. J. Ives (ed.), *The Scientific Basis of Filtration,* Noordhoff, Leyden, 1975.
2. B. F. Ruth, *Ind. Eng. Chem., 38,* 564 (1946).
3. M. Shirato, T. Aragaki, and E. Iritani, *J. Chem. Eng. Jpn., 13,* 61 (1980).
4. M. Shirato, T. Murase, and K. Atsumi, *Int. Chem. Eng., 22,* 689 (1982).
5. F. M. Tiller, *Filtr. Sep., 12,* 386 (1975).
6. F. M. Tiller (ed.), *Theory and Practice of Solid–Liquid Separation,* University of Houston, Houston, 1978.
7. R. E. Gibson, G. L. England, and M. J. L. Hussey, *Geotechnique, 17,* 261 (1967).
8. R. E. Gibson, R. L. Schiffman, and K. W. Cargill, *Can. Geotech. J., 18,* 280 (1981).
9. A. McNabb, *Q. Appl. Math., 17,* 337 (1960).
10. K. Terzaghi, *Theoretical Soil Mechanics,* Wiley, New York, 1942.
11. D. Znidarcic and R. L. Schiffman, *Geotechnique, 32,* 387 (1982).

12. J. R. Philip, *Aust. J. Soil Res.*, *6*, 249 (1968).
13. J. R. Philip and D. E. Smiles, *Ibid.*, *7*, 1 (1969).
14. D. E. Smiles and M. J. Rosenthal, *Ibid.*, *6*, 237 (1968).
15. D. E. Smiles, *Chem. Eng. Sci.*, *25*, 985 (1970).
16. D. E. Smiles, *Ibid.*, *33*, 1355 (1978).
17. D. E. Smiles, P. A. C. Raats, and J. H. Knight, *Ibid.*, *37*, 707 (1982).
18. R. J. Wakeman, *Trans. Inst. Chem. Eng.*, *56*, 258 (1978).
19. D. Zaslavsky, *Soil Sci.*, *98*, 317 (1964).
20. P. C. Carman, *Trans. Inst. Chem. Eng.*, *15*, 150 (1937).
21. C. A. Black (ed.), *Methods of Soil Analysis*, Part 1, American Society of Agronomy, Madison, Wisconsin, 1965.
22. D. Croney, J. D. Coleman, and P. M. Bridge, *Road Res. Tech. Pap.*, p. 24 (1952).
23. J. R. Philip, *Aust. J. Soil Res.*, *7*, 99 (1969).
24. G. D. Aitchison (ed.), *Pore Pressure and Suction in Soils*, Butterworths, London, 1961.
25. M. Shirato, H. Kato, K. Kobayashi, and H. Sakezaki, *J. Chem. Eng. Jpn.*, *3*, 98 (1970).
26. E. C. Childs, *The Physical Basis of Soil Water Phenomena*, Wiley-Interscience, London, 1969.
27. J. R. Philip, *Annu. Rev. Fluid Mech.*, *2*, 177 (1970).
28. J. R. Philip, *Adv. Hydrosci.*, *5*, 215 (1969).
29. J. R. Philip, *Soil Sci.*, *116*, 328 (1973).
30. J. H. Knight and J. R. Philip, *Ibid.*, *116*, 407 (1973).
31. J.-Y. Parlange, *Ibid.*, *111*, 134 (1971).
32. D. E. Smiles and A. G. Harvey, *Ibid.*, *116*, 391 (1973).
33. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd ed., Oxford, London, 1959.
34. D. E. Smiles, *Soil Sci.*, *117*, 140 (1974).
35. D. E. Smiles, *Aust. J. Soil Res.*, *14*, 389 (1976).
36. J. R. Philip and D. E. Smiles, *Adv. Colloid Interface Sci.*, *17*, 83 (1982).
37. B. E. Clothier, J. H. Knight, and I. White, *Soil Sci.*, *132*, 252 (1981).
38. J. R. Philip, *Ibid.*, *117*, 257 (1974).
39. P. A. C. Raats, in *Rational Thermodynamics*, 2nd ed. (C. Truesdell, ed.), Springer-Verlag, New York, 1984, Appendix 5D.

Received by editor July 29, 1986